MATH 4030 Differential Geometry
Tutorial 1, 13 September 2017

1. Reparametrize the following curves by arc length.

   (a) $\alpha : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (r\cos kt, r\sin kt),\ \ k, r > 0$
   - $\alpha'(t) = (-rk\sin kt, rk\cos kt)$
   - $|\alpha'(t)| = rk$
   - $s(t) = \int_0^t rk\ dt = rkt$
   - $t(s) = \frac{s}{rk}$
   - $\therefore\ \beta(s) = (r\cos(\frac{s}{r}), r\sin(\frac{s}{r})),\ \ s \in \mathbb{R}$ is a reparametrization of α by arc length.

   (b) $\alpha : \mathbb{R} \to \mathbb{R}^3 : t \mapsto (a\cos t, a\sin t, bt)$
   - $\alpha'(t) = (-a\sin t, a\cos t, b)$
   - $|\alpha'(t)| = \sqrt{a^2+b^2}$
   - $s(t) = \int_0^t \sqrt{a^2+b^2}\ dt = \sqrt{a^2+b^2}t$
   - $t(s) = \frac{s}{\sqrt{a^2+b^2}}$
   - $$\therefore \beta(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right),\ \ s \in \mathbb{R}$$
     is a reparametrization of α by arc length.

2. (HW 1 Suggested problem Q8)

($\Longrightarrow$) $|\alpha(t)| = c,\ c$ is a nonzero constant

$$\begin{aligned}
&\Rightarrow\ \langle\alpha(t), \alpha(t)\rangle = |\alpha(t)|^2 = c^2 \\
&\Rightarrow\ 2\langle\alpha(t), \alpha'(t)\rangle = \langle\alpha'(t), \alpha(t)\rangle + \langle\alpha(t), \alpha'(t)\rangle = \tfrac{d}{dt}\langle\alpha(t), \alpha(t)\rangle = \tfrac{d}{dt}(c^2) = 0 \\
&\Rightarrow\ \langle\alpha(t), \alpha'(t)\rangle = 0.
\end{aligned}$$

($\Longleftarrow$) Let $f(t) = \langle\alpha(t), \alpha(t)\rangle$. Then by the above calculation $f'(t) \equiv 0$. Since I is connected, it follows that $f(t)$ is a constant, and so is $\sqrt{f(t)} = |\alpha(t)|$. This constant cannot be zero otherwise α is not regular.

3. (HW 1 Suggested problem Q9) Write $\alpha(t) = (x(t), y(t)),\ t \in \mathbb{R}$. Let P be the point on the circumference of the disk, and Q be the center of the disk. For simplicity, assume that $P = O$ the origin when $t = 0$, and that the velocity of the disk is 1 whose direction is the positive x-axis. Then t is the horizontal distance travelled by Q at time t which is also the angle that P rotates (clockwise) about Q. We get
$$\begin{cases} x(t) = & -\sin t + t \\ y(t) = & 1 - \cos t \end{cases}.$$
Now $\alpha'(t) = (-\cos t + 1, \sin t)$ which is zero if and only if $t = 0, \pm 2\pi, \pm 4\pi, \ldots$. In other words, this curve is not regular exactly when P touches the x-axis. Finally, the length of the part of α corresponding to a complete rotation of the disk is equal to
$$\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\ dt = 8.$$